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Field theoretic treatment of the strong-coupling attractive Hubbard model with diagonal disorder

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Abstract. We present a field theoretic formulation of the strong-coupling attractive Hubbard model with Gaussian distributed random on-site energies. Self-consistency equations for the order parameters of superconductivity and charge ordering are derived at lowest loop order and solved for the limiting cases of weak and strong disorder. We also study dynamical properties and calculate the influence of disorder on the Bogoliubov–Anderson mode in the superconducting phase.

1. Introduction

The problem of electronic systems with attractive local interaction has attracted considerable interest in recent years [1]. In these systems strong-coupling leads to the formation of electron pairs which may undergo a Bose-condensation-type transition to a superconducting phase. Strong-coupling Hubbard models with attractive on-site interaction and delta-correlated disorder have been devised as the natural candidates for superconducting glasses [2].

In the present paper we study the large-negative-U (LNU) Hubbard model with random on-site energies obeying a Gaussian distribution. This disordered model is equivalent to an anisotropic pseudospin quantum Heisenberg model in a random external field and has been the subject of some discussion [2,3] where mean-field-like operator decoupling schemes and more simple probability distributions were used. The random field problem, as such, has been approached for classical spin systems by various authors [4–6], while a field theoretic treatment of this problem appeared to be hampered by the lack of a simple representation of spin operators allowing for full control of quantum fluctuations. This goal can be facilitated by a Grassmannian representation of the pseudospin operators based on the diagram technique proposed by Popov and Fedotov [7] which leads to a semionic theory of the LNU Hubbard model [8]. This technique utilizes the observation that the constraint imposed on the Hilbert space of a fermionic representation of spin-$\frac{1}{2}$ operators can be modelled correctly by a modified Matsubara frequency spectrum with values interpolating between the bosonic and fermionic case. The Hilbert space constraint showing up in the strong-coupling limit of the attractive Hubbard model is then correctly reproduced by a modification of the original formulation resulting in spin-dependent semionic Matsubara frequencies [8].

We find that the results obtained by Micnas *et al* [3] applying a mean-field approximation for the square and two-delta distribution are formally equivalent to the saddle-point solution of our field theoretic approach and can be naturally extended to Gaussian disorder.

An explicit solution of the self-consistency equations is given for the cases of weak and strong disorder. We report results for the critical temperatures, the superconducting order parameter and the density of states. Moreover, the effect of disorder on dynamical properties is investigated by calculating the Bogoliubov–Anderson mode in the superconducting phase.

2. Field theoretic derivation of self-consistency equations

We consider the Hubbard Hamiltonian with hopping amplitude t between neighbouring sites and attractive on-site interaction $U > 0$. Diagonal disorder is introduced by random on-site energies ϵ_i.

$$\mathcal{H} = t \sum_{\langle ij\rangle\sigma} a^{+}_{i\sigma} a_{j\sigma} - U \sum_{i} n_{i\uparrow} n_{i\downarrow} - \frac{1}{2} \sum_{i\sigma} (\mu + \epsilon_i) n_{i\sigma}. \tag{2.1}$$

In the case of strong-coupling $U \gg t$ a well known mapping of the attractive Hubbard Hamiltonian leads to an effective model equivalent to a pseudospin anisotropic quantum Heisenberg Hamiltonian with attractive interaction in the XY-sector and repulsion in the Ising sector

$$\mathcal{H} = J \sum_{\langle ij\rangle} (-A^{+}_i A_j + N_i N_j) - \sum_{i} (\mu + \epsilon_i) N_i \tag{2.2}$$

where $J = 2t^2/U$ and pair operators $A_i = a_{i\downarrow} a_{i\uparrow}$ and $A^{+}_i = a^{+}_{i\uparrow} a^{+}_{i\downarrow}$ have been introduced. We further apply the identification $N_i = A^{+}_i A_i = \frac{1}{2}(n_{i\uparrow} + n_{i\downarrow})$ which is valid in the restricted Hilbert space of the strong-coupling limit where only doubly occupied or empty sites exist. The random on-site energies ϵ_i are assumed to be only locally correlated (site-diagonal) and to obey a Gaussian distribution with zero mean

$$P(\epsilon_i) = \frac{1}{w\sqrt{2\pi}} \exp\left(- \frac{\epsilon_i^2}{2w^2} \right) \qquad \text{with } \langle\!\langle \epsilon_i \rangle\!\rangle = 0 \text{ and } \langle\!\langle \epsilon_i \epsilon_j \rangle\!\rangle = w^2 \delta_{ij}. \tag{2.3}$$

The quenched average of the partition function may then be written as a functional integral over replicated Grassmann fields

$$\langle\!\langle Z^{N_{\mathrm{R}}} \rangle\!\rangle = \int \prod_{i\sigma\alpha\tau} \mathcal{D}\overline{\psi}^{\alpha}_{i\sigma}(\tau) \mathcal{D}\psi^{\alpha}_{i\sigma}(\tau) \int \mathrm{d}\epsilon_i \, P(\epsilon_i) \exp \mathcal{S}(\overline{\psi}^{\alpha}_{i\sigma}(\tau), \psi^{\alpha}_{i\sigma}(\tau)) \tag{2.4}$$

with replica index $\alpha = 1, \ldots, N_{\mathrm{R}}$ and the action

$$\begin{aligned} \mathcal{S}(\overline{\psi}^{\alpha}_{i\sigma}(\tau), \psi^{\alpha}_{i\sigma}(\tau)) = \int_0^{\beta\hbar} \mathrm{d}\tau \Big(& \sum_{i\sigma\alpha} \overline{\psi}^{\alpha}_{i\sigma}(\tau) \left(\partial_\tau + \mu/2\hbar + \epsilon_i/2\hbar\right) \psi^{\alpha}_{i\sigma}(\tau) \\ & + J \sum_{\langle ij\rangle\alpha} \Big(\overline{\psi}^{\alpha}_{i\uparrow}(\tau) \overline{\psi}^{\alpha}_{i\downarrow}(\tau) \psi^{\alpha}_{j\downarrow}(\tau) \psi^{\alpha}_{j\uparrow}(\tau) \\ & - \frac{1}{4} \sum_{\sigma\sigma'} \overline{\psi}^{\alpha}_{i\sigma}(\tau) \psi^{\alpha}_{i\sigma}(\tau) \overline{\psi}^{\alpha}_{j\sigma'}(\tau) \psi^{\alpha}_{j\sigma'}(\tau) \Big) \Big). \end{aligned} \tag{2.5}$$

The Gaussian distribution of the random energies can be integrated out and yields the effective action in Matsubara frequency space

$$\begin{aligned}\mathcal{S}(\overline{\psi}_{i\sigma}^{\alpha n},\psi_{i\sigma}^{\alpha n}) &= \sum_{i\sigma\alpha n}\overline{\psi}_{i\sigma}^{\alpha n}\,(\mathrm{i}\beta\hbar z_{n\sigma}+\beta\mu/2)\,\psi_{i\sigma}^{\alpha n}\\ &\quad+\beta J\sum_{(ij)\alpha n_1 n_2\omega}\Big(\overline{\psi}_{i\uparrow}^{\alpha n_1}\overline{\psi}_{i\downarrow}^{\alpha-n_1+\omega}\psi_{j\downarrow}^{\alpha n_2}\psi_{j\uparrow}^{\alpha-n_2+\omega}\\ &\quad-\frac{1}{4}\sum_{\sigma\sigma'}\overline{\psi}_{i\sigma}^{\alpha n_1}\psi_{i\sigma}^{\alpha n_1+\omega}\overline{\psi}_{j\sigma'}^{\alpha n_2}\psi_{j\sigma'}^{\alpha n_2-\omega}\Big)\\ &\quad+\frac{1}{8}(\beta w)^2\sum_{i\sigma\sigma'\alpha\beta n_1 n_2}\overline{\psi}_{i\sigma}^{\alpha n_1}\psi_{i\sigma}^{\alpha n_1}\overline{\psi}_{i\sigma'}^{\beta n_2}\psi_{i\sigma'}^{\beta n_2}.\end{aligned}\tag{2.6}$$

The (semionic) Matsubara frequency spectrum $z_{n\sigma}=(\pi/\beta\hbar)(2n+1+\sigma/2)$ has spin-dependent modified values as a consequence of the proper treatment of the Hilbert space constraint. This is achieved by introducing a special valued imaginary magnetic field which preserves time-reversal invariance and projects out contributions from unphysical states in the partition function and those observables or correlation functions which preserve the global SU(2) pseudospin symmetry of Hamiltonian (2.2) (see [8] for further details).

These modified frequencies are related to a global U(1) gauge transformation of the original Grassmann fields which acquire an additional phase factor $\exp((\mathrm{i}\pi\sigma/2\beta\hbar)\tau)$ resulting in unusual periodicity conditions $\psi_\sigma(\beta\hbar)=\mathrm{i}\sigma\psi_\sigma(0)$ of the anticommuting fields.

In order to extract physically relevant properties from this form of the action we have to decouple the products of four Grassmann fields by introducing $2N_\mathrm{R}+1$ local Hubbard–Stratonovich fields $\alpha_i^\alpha, x_i^\alpha, y_i$ which correspond to the superconducting order parameter, the pair occupation number or (pseudo)magnetization, and a Bose field which will serve as an integration variable reflecting the Gaussian probability distribution, respectively.

The action then takes the form

$$\begin{aligned}\mathcal{S}(\overline{\psi}_{i\sigma}^{\alpha n},\psi_{i\sigma}^{\alpha n}) &= -\tfrac{1}{2}\sum_i y_i^2+\beta\sum_{(ij)\alpha\omega}(J^{-1})_{ij}\left(-x_i^{\alpha\omega}x_j^{\alpha-\omega}+\overline{\alpha}_i^{\alpha\omega}\alpha_j^{\alpha\omega}\right)\\ &\quad+\sum_{i\alpha\sigma n\omega}\overline{\psi}_{i\sigma}^{\alpha n}\left((\mathrm{i}\beta\hbar z_{n\sigma}+\beta\mu/2+\tfrac{1}{2}\beta w y_i)\delta_{\omega 0}+\beta x_i^{\alpha-\omega}\right)\psi_{i\sigma}^{\alpha n+\omega}\\ &\quad-\beta\sum_{i\alpha n\omega}\left(\overline{\alpha}_i^{\alpha\omega}\psi_{i\downarrow}^{\alpha n}\psi_{i\uparrow}^{\alpha-n+\omega}+\alpha_i^{\alpha\omega}\overline{\psi}_{i\uparrow}^{\alpha n}\overline{\psi}_{i\downarrow}^{\alpha-n+\omega}\right).\end{aligned}\tag{2.7}$$

If we adopt the two-component spinor notation of Popov [10]

$$\overline{\chi}_i^n=\left(\overline{\psi}_{i\uparrow}^n,\psi_{i\downarrow}^{-n}\right)\qquad \chi_i^n=\begin{pmatrix}\psi_{i\uparrow}^n\\ -\overline{\psi}_{i\downarrow}^{-n}\end{pmatrix}\tag{2.8}$$

action (2.7) may be written in a compact form bilinear in the fermion fields

$$\mathcal{S}(\overline{\chi}_i^{\alpha n},\chi_i^{\alpha n})=-\tfrac{1}{2}\sum_i y_i^2+\beta\sum_{(ij)\alpha}(J^{-1})_{ij}\left(-x_i^\alpha x_j^\alpha+\overline{\alpha}_i^\alpha\alpha_j^\alpha\right)+\sum_{i\alpha n}\overline{\chi}_i^{\alpha n}A_i^{\alpha n}\chi_i^{\alpha n}\tag{2.9}$$

with the matrix $\mathbf{A}$ given by

$$(A_i^{\alpha n}) = \begin{pmatrix} \mathrm{i}\beta\hbar z_{n\uparrow} + \beta\mu/2 + \beta x_i^\alpha + \frac{1}{2}\beta w y_i & \beta\alpha_i^\alpha \\ -\beta\overline{\alpha_i^\alpha} & -\mathrm{i}\beta\hbar z_{n\downarrow} + \beta\mu/2 + \beta x_i^\alpha + \frac{1}{2}\beta w y_i \end{pmatrix}. \tag{2.10}$$

Here the static approximation $\alpha_i^{\alpha\omega} = \alpha_i^\alpha \delta_{\omega 0}$ and $x_i^{\alpha\omega} = x_i^\alpha \delta_{\omega 0}$ has been applied, which neglects the coupling between different energies of the fermionic fields and allows identification of the saddle-point solution with the usual mean-field operator decoupling schemes.

The remaining conventional integration over the spinor Grassmann fields can now be carried out. As a result one ends up with the (divergent) fermion determinant

$$\ln \det\mathbf{A} = \tfrac{1}{2} \sum_{\epsilon=\pm 1} \ln \prod_n \beta \left((\hbar z_{n\sigma})^2 + |\alpha|^2 + \left(\mu/2 + \tfrac{1}{2} w y + \epsilon x\right)^2 \right) \tag{2.11}$$

which can be regularized without changing the physical results by $\mathbf{A} \to \mathbf{A}' = \mathbf{A}_0^{-1}\mathbf{A}$ with $\mathbf{A}_0 = \mathbf{A}(\mu = \alpha = x = w = 0)$. One then has

$$\ln \det\mathbf{A}' = \tfrac{1}{2} \sum_{\epsilon=\pm 1} \ln\cosh \beta \sqrt{|\alpha|^2 + \left(\mu/2 + \tfrac{1}{2} w y + \epsilon x\right)^2}. \tag{2.12}$$

We have made the usual assumption that α_i^α and x_i^α are independent of the replica index. We have also set $\alpha_i = \alpha$ and $x_i = (-1)^i x$ corresponding to the homogeneous singlet superconducting order (SC) and charge ordering (CDW) on two interpenetrating sublattices, respectively. This choice also ensures convergence of the Gaussian integrals over the decoupling fields. Performing the replica limit $N_\mathrm{R} \to 0$ and taking $W = zJ$, f, the free energy per site is readily derived

$$f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}y \mathrm{e}^{-\frac{1}{2}y^2} \left(\frac{1}{W}(x^2 + |\alpha|^2) - \tfrac{1}{2} kT \sum_{\epsilon=\pm 1} \ln\cosh \beta \sqrt{|\alpha|^2 + \left(\mu/2 + \tfrac{1}{2} w y + \epsilon x\right)^2} \right). \tag{2.13}$$

The saddle-point solutions for α and x are then obtained by the conditions

$$\partial f/\partial\overline{\alpha} = 0 = \partial f/\partial x \tag{2.14}$$

which finally lead to the self-consistency equations for the order parameters of superconductivity

$$1 = \frac{W}{4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}y \mathrm{e}^{-\frac{1}{2}y^2} \sum_{\epsilon=\pm 1} \frac{\tanh \beta \sqrt{|\alpha|^2 + (\mu/2 + \frac{1}{2} w y + \epsilon x)^2}}{\sqrt{|\alpha|^2 + (\mu/2 + \frac{1}{2} w y + \epsilon x)^2}} \tag{2.15}$$

and charge ordering

$$x = \frac{W}{4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}y \mathrm{e}^{-\frac{1}{2}y^2} \sum_{\epsilon=\pm 1} \left(x + \frac{\epsilon}{2}(\mu + w y) \right) \times \frac{\tanh \beta \sqrt{|\alpha|^2 + (\mu/2 + \frac{1}{2} w y + \epsilon x)^2}}{\sqrt{|\alpha|^2 + (\mu/2 + \frac{1}{2} w y + \epsilon x)^2}}. \tag{2.16}$$

Similar equations for a square and a two-delta distribution were derived by Micnas *et al* [3] applying a mean-field-like decoupling of operators. Note that (2.15) is valid only for non-vanishing superconducting order parameter, while it implies vanishing x for finite w and half band filling. Therefore, the saddle-point solution in the presence of disorder does not allow for coexistence of a superconducting and a charge ordered phase at half filling in agreement with the conclusions reached in [3].

Our field theoretic approach also permits calculation of the fermionic one-particle propagators at the saddle point, which are given by

$$\begin{aligned}\mathcal{G}_i(z_{n\sigma}) &= \langle\!\langle G_i(z_{n\sigma})\rangle\!\rangle \\ &= \frac{\hbar}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}y \mathrm{e}^{-\frac{1}{2}y^2} \frac{-\mathrm{i}\hbar z_{n\sigma} + \mu/2 + x_i + \frac{1}{2}wy}{(\hbar z_{n\sigma})^2 + |\alpha|^2 + (\mu/2 + \frac{1}{2}wy + x_i)^2}\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\mathcal{F}_i(z_{n\sigma}) &= \langle\!\langle F_i(z_{n\sigma})\rangle\!\rangle \\ &= \frac{\hbar}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}y \mathrm{e}^{-\frac{1}{2}y^2} \frac{\alpha}{(\hbar z_{n\sigma})^2 + |\alpha|^2 + (\mu/2 + \frac{1}{2}wy + x_i)^2}\end{aligned} \tag{2.18}$$

where the Gaussian averages are performed over the zero-loop propagators $G_i(z_{n\sigma})$ and $F_i(z_{n\sigma})$ of the clean LNU Hubbard model [8] with the chemical potential being modified by the local charge and on-site energy. The propagators are totally local as a consequence of the local U(1) invariance of the action (2.5) under $\psi_{i\sigma} \to \mathrm{e}^{\mathrm{i}\varphi_{i\sigma}}\psi_{i\sigma}$ with $\varphi_{i\sigma} = -\varphi_{i-\sigma}$.

From (2.17) a further equation determining the chemical potential can be obtained

$$\begin{aligned}\nu = \tfrac{1}{2}N\beta\hbar\sum_{in\sigma}\mathcal{G}_i(z_{n\sigma}) &= \tfrac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{-\frac{1}{2}y^2}\sum_{\epsilon=\pm1}\frac{1}{8}(\mu + wy + 2\epsilon x) \\ &\quad\times\frac{\tanh\beta\sqrt{|\alpha|^2 + (\mu/2 + \frac{1}{2}wy + \epsilon x)^2}}{\sqrt{|\alpha|^2 + (\mu/2 + \frac{1}{2}wy + \epsilon x)^2}}.\end{aligned} \tag{2.19}$$

In particular one has $\mu = 0$ at half filling ($\nu = \frac{1}{2}$) for arbitrary temperature and strength of disorder.

3. Discussion of mean-field results

From equations (2.15) and (2.16) conditions for the critical temperatures of a transition to a superconducting or charge-ordered pure phase can be derived. For general band filling one has

$$1 = \frac{\beta_c W}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{-\frac{1}{2}y^2}\frac{\tanh\frac{1}{2}\beta_c(\mu + wy)}{\frac{1}{2}\beta_c(\mu + wy)} \qquad \text{for SC} \tag{3.1}$$

and

$$1 = \frac{\beta_c W}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{-\frac{1}{2}y^2}\frac{1}{\cosh^2\frac{1}{2}\beta_c(\mu + wy)} \qquad \text{for CDW.} \tag{3.2}$$

One easily recognizes that the inequality $T_c^{\mathrm{SC}} \geqslant T_c^{\mathrm{CDW}}$ holds, becoming an equality for half filling in the clean limit $w = 0$ where the XY-sector and Ising sector of the pseudospin model (2.2) become equivalent due to a dual symmetry (or partial particle–hole transformation). This symmetry maps the attractive Hubbard model onto the repulsive one which is equivalent to an isotropic Heisenberg model in the strong-coupling limit. Note that in this case (2.15) and (2.16) coincide, resulting in a single equation for the modulus of the total (pseudo)magnetization with its orientation being undetermined as a consequence of O(3) symmetry.

The pure superconducting phase is always favoured in the presence of even arbitrarily weak disorder. One can further show that the pure superconducting state does not exhibit an instability towards formation of charge ordering. Coexistence of superconductivity and charge ordering is thus only possible for half filling and in the clean limit, while in all other cases a transition to a pure superconducting phase occurs.

For general filling, but vanishing disorder, the well known results for the clean LNU Hubbard model [2,8,9] are retrieved.

At half-band filling and for small disorder parameter $\delta = w/W \ll 1$ the self-consistency equations can be explicitly evaluated and neglecting terms of order $\mathcal{O}(\delta^4)$ or higher one has the results for the critical temperatures

$$kT_c^{\mathrm{SC}} = (W/2)(1 - \tfrac{1}{3}\delta^2) \tag{3.3}$$

$$kT_c^{\mathrm{CDW}} = (W/2)(1 - \delta^2) \tag{3.4}$$

and the low-temperature value of the SC order parameter

$$|\alpha| = (W/2)(1 - \tfrac{1}{2}\delta^2) \qquad \text{for} \qquad T \to 0. \tag{3.5}$$

In the case of large disorder parameter $\delta \gg 1$ in this temperature regime one obtains

$$|\alpha| \sim \delta\, \mathrm{e}^{-\delta\sqrt{\pi/2}} \tag{3.6}$$

indicating a strong suppression of superconducting order in the dirty limit. However, in the saddle-point solution superconductivity persists up to arbitrarily strong disorder.

The critical behaviour of the order parameter near the transition point is given by the usual mean-field exponent

$$|\alpha|^2 = \frac{T_c - T}{T_c} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}y \mathrm{e}^{-\frac{1}{2}y^2} \times \frac{\frac{1}{2}(\mu + wy)^2}{\sinh[\beta_c(\mu + wy)]/[\beta_c(\mu + wy)] - 1} \qquad \text{for } T_c - T \ll T_c. \tag{3.7}$$

The numerical prefactor may be determined for half filling and near the clean limit

$$|\alpha|^2 = \tfrac{3}{4}(T_c - T)/T_c(1 - \tfrac{1}{5}\delta^2). \tag{3.8}$$

Using the conventional relation

$$\rho(\varepsilon) = \mp(1/\pi) \lim_{\eta \to 0} \mathrm{Im}\, \mathcal{G}(\varepsilon \pm \mathrm{i}\eta) \tag{3.9}$$

and the result (2.17) for the normal propagator the density of states is then easily determined

$$\rho(\varepsilon) = \theta(|\varepsilon| - |\alpha|) \frac{4}{w\sqrt{2\pi}} \exp\left(-\frac{2}{w^2}\left(\varepsilon^2 + \left(\frac{\mu}{2}\right)^2 - |\alpha|^2\right)\right) \times \left(\frac{|\varepsilon|}{\sqrt{\varepsilon^2 - |\alpha|^2}} \cosh\left(\frac{2}{w^2}\mu\sqrt{\varepsilon^2 - |\alpha|^2}\right) - \operatorname{sgn}\varepsilon\, \sinh\left(\frac{2}{w^2}\mu\sqrt{\varepsilon^2 - |\alpha|^2}\right)\right). \tag{3.10}$$

For half filling this result reduces to a characteristic square-root singularity behaviour at $\pm|\alpha|$ with an additional Gaussian damping factor due to disorder. This is compared to the expression for the clean case

$$\rho(\varepsilon) = \tfrac{1}{2}(1-\mu/2\varepsilon)\left(\delta\left(\varepsilon-\sqrt{|\alpha|^2+(\mu/2)^2}\right)+\delta\left(\varepsilon+\sqrt{|\alpha|^2+(\mu/2)^2}\right)\right) \tag{3.11}$$

which is equivalent to the usual BCS result [11] for electrons with infinite mass, reflecting the locality of the propagators.

4. Bogoliubov–Anderson mode

In order to calculate the excitation spectra in the pure superconducting phase, the full time dependence in action (2.5) has to be retained and the fluctuations around the saddle-point value of the superconducting order parameter α_0 are separated by

$$\alpha(\tau) = \alpha_0 + \phi(\tau). \tag{4.1}$$

The part of the action describing the fluctuations is then given by

$$\mathcal{S}_{\mathrm{fl}} = \frac{1}{\hbar}\int_0^{\beta\hbar} \mathrm{d}\tau \left(-\frac{1}{W_q}\left(\overline{\alpha}_0\phi(\tau)+\alpha_0\overline{\phi}(\tau)+\overline{\phi}(\tau)\phi(\tau)\right)+\ln\det\left(\mathbf{1}+\mathbf{A}_{\mathrm{sp}}^{-1}\,u_\phi\right)\right) \tag{4.2}$$

where $\mathbf{A}_{\mathrm{sp}}$ is the saddle-point value of matrix (2.10) and

$$\mathbf{u}_\phi = \begin{pmatrix} 0 & \phi(\tau) \\ -\overline{\phi}(\tau) & 0 \end{pmatrix}. \tag{4.3}$$

For simplicity of notation replica indices have been suppressed.

Expanding (4.2) up to second order in $\phi, \overline{\phi}$ and restricting ourselves to infinitesimal phase fluctuations of $\alpha(\tau) = \alpha_0 \mathrm{e}^{\mathrm{i}\varphi(\tau)}$ one has $\phi(\tau) = \mathrm{i}\alpha_0\varphi(\tau)$ and the fluctuation part of the action in frequency space is readily derived

$$\mathcal{S}_{\mathrm{fl}} = \tfrac{1}{2}\beta|\alpha_0|^2\sum_\omega \overline{\xi}_\omega \mathbf{\Pi}_\omega \xi_\omega \tag{4.4}$$

where $\overline{\xi}_\omega = (\overline{\varphi}_\omega, \varphi_\omega)$ and the kernel of the bilinear form in (4.4) is given by the inverse two-particle propagator matrix

$$\mathbf{\Pi}_\omega = \begin{pmatrix} (1/\hbar)\Pi_{GG}(\omega)-1/W_q & (1/\hbar)\Pi_{FF}(\omega) \\ (1/\hbar)\Pi^*_{FF}(\omega) & (1/\hbar)\Pi_{GG}(-\omega)-1/W_q \end{pmatrix}. \tag{4.5}$$

The matrix elements are

$$\Pi_{GG}(\omega) = \frac{1}{\beta\hbar}\sum_n G(z_{n\sigma})G(-z_{n\sigma}+\omega) \tag{4.6}$$

$$\Pi_{FF}(\omega) = \frac{1}{\beta\hbar}\sum_n F(z_{n\sigma})F(-z_{n\sigma}+\omega) \tag{4.7}$$

which still depend on the integration variable y since averaging has not been performed yet. In the replica limit $N_{\mathrm{R}} \to 0$ the inverse of the two-particle propagator reads

$$\Pi(q,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}y \mathrm{e}^{-\frac{1}{2}y^2} \begin{pmatrix} \Pi_{GG}(\omega) - \hbar/W_q & \Pi_{FF}(\omega) \\ \Pi^{*}_{FF}(\omega) & \Pi_{GG}(-\omega) - \hbar/W_q \end{pmatrix}. \tag{4.8}$$

The excitation spectrum is now derived from the condition that the bilinear form of the fluctuation fields becomes singular, i.e. $\det \Pi(q,\omega) = 0$ and

$$|1 - (W_q/\hbar)\langle\langle \Pi_{GG}(\omega)\rangle\rangle| = \pm|(W_q/\hbar)\langle\langle \Pi_{FF}(\omega)\rangle\rangle|. \tag{4.9}$$

The different signs correspond to the two excitation branches related to longitudinal and transversal fluctuations of the order parameter around the saddle-point value. The vanishing mass term of this equation reproduces the self-consistency condition (2.15) for the superconducting order parameter as a consequence of the soft-mode behaviour of the Bogoliubov–Anderson mode. We recover the linear dispersion relation

$$\hbar\omega(q) = v\sqrt{(2/d)}\, qa \tag{4.10}$$

for the sound-like compressional mode of superfluid pairs with the velocity v given by

$$\frac{1}{v^2} = \left\langle\!\!\left\langle \frac{X_y}{R_y^2} \right\rangle\!\!\right\rangle + \frac{\langle\langle B_y X_y / R_y^2 \rangle\rangle^2}{1 - \langle\langle B_y^2 X_y / R_y^2 \rangle\rangle} \tag{4.11}$$

where we have introduced the abbreviations

$$B_y = \tfrac{1}{2}(\mu + wy) \qquad R_y = \sqrt{|\alpha|^2 + B_y^2} \qquad X_y = \frac{W}{2}\frac{\tanh \beta R_y}{R_y}. \tag{4.12}$$

In the clean limit $X_y \equiv 1$, and the result $v = |\alpha|$ with the value of the exact order parameter as suggested in [8] is retrieved. It is apparent from (4.11) that disorder leads to a damping of the mode in addition to the effect caused by the decrease of the order parameter. For low temperatures and weak disorder we find at half filling

$$v = (W/2)(1 - \tfrac{3}{4}\delta^2) \tag{4.13}$$

where the saddle-point value (3.5) has been used and the disorder-induced slowing down of the collective motion of Bose-condensed pairs is explicit.

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